# The Euclidean Distance Degree

in Machine Learning

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# Outline

Motivation and Definition

The Euclidean Distance Degree in Machine Learning

The Euclidean Distance Degree and Duality

# Motivation and Definition

# Disclaimer

This line of inquiry was initiated by

- Jan Draisma (University of Bern),
- Emil Horobeț (Sapientia Hungarian University of Transylvania),
- Giorgio Ottaviani (University of Florence),
- Bernd Sturmfels (UC Berkley), and
- Rekha Thomas (University of Washington)

in [Draisma et al., 2016].

Its roots date back to Roger Cotes.

# Setup I

All feasible objectives are parametrized by an algebraic map

$$f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$
$$\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \longmapsto \begin{pmatrix} f_1(a_1, \ldots, a_m) \\ \vdots \\ f_n(a_1, \ldots, a_m) \end{pmatrix},$$

with polynomials $f_1, \ldots, f_n \in \mathbb{R}[x_1, \ldots, x_m]$.

We complexify the map $f_{\mathbb{C}} : \mathbb{C}^m \to \mathbb{C}^n$.

The closure of $\operatorname{img}(f_{\mathbb{C}})$ is an algebraic set $X$ (e.g. by [Serre, 1956, p. 12]).

# Example: Rank-1-Matrices

$$\mathbb{R}^{2\times 1} \times \mathbb{R}^{1\times 2} \longrightarrow \mathbb{R}^{2\times 2}$$
$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 & y_2 \end{pmatrix}\right) \longmapsto \begin{pmatrix} x_1y_1 & x_1y_2 \\ x_2y_1 & x_2y_2 \end{pmatrix}$$

Here $X$ is the determinantal variety $Z(x_{11}x_{22} - x_{12}x_{21}) \subsetneq \mathbb{C}^{2\times 2}$.

# Setup II

We consider points $\overrightarrow{u} \in \mathbb{C}^n$, and analyse the non-singular critical points of

$$d_{\overrightarrow{u}} : X \longrightarrow \mathbb{C}$$
$$\overrightarrow{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \sum_{i=1}^{n} (u_i - x_i)^2.$$

The *Euclidean Distance Degree* $\mathrm{EDdeg}(X)$ of $X$
counts the number of non-singular critical points of $d_{\overrightarrow{u}}$ for a general $\overrightarrow{u} \in \mathbb{C}^n$.

# Example: Cardioid

Find the non-singular critical points on the Cardioid curve:



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A point $\overrightarrow{x} \in X \setminus X_{\text{sing}}$ is critical $\iff \overrightarrow{u} - \overrightarrow{x} \perp T_{\overrightarrow{x}} X$.

# How To: Find $\mathrm{EDdeg}(X)$ for the Cardioid Algebraically?

We start with $X = Z(f)$, for $f := (x_1^2 + x_2^2 + x_2)^2 - (x_1^2 + x_2^2) \in \mathbb{C}[x_1, x_2]$.

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4. Inside $\mathbb{C}^n_{\overrightarrow{x}} \times \mathbb{C}^n_{\overrightarrow{u}}$: Determine the pairs $(\overrightarrow{x}, \overrightarrow{u})$ for which $x \in X$ and $\overrightarrow{u} - \overrightarrow{x} \in (T_{\overrightarrow{x}} X)^\perp$.

$$I_{\text{pairs}} := I_X + \left( \det \begin{pmatrix} u_1 - x_1 & u_2 - x_2 \\ \partial_{x_1} f & \partial_{x_2} f \end{pmatrix} \right).$$

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6. The degree $Z(I_{\text{crit}})$ is EDdeg($X$).

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# Bonus: How to find the Evolute?

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# The Euclidean Distance Degree in Machine Learning

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# The Eckart-Young Theorem

Fix dimensions $m, n \in \mathbb{N}$ and some rank $r$. We assume w.l.o.g. $r \leq m \leq n$.

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Let $U \in \mathbb{R}^{m\times n}$ be a fixed matrix, with SVD

$$U = T_1 \cdot \mathrm{diag}(\sigma_1, \ldots, \sigma_m) \cdot T_2, \quad \text{with } T_1 \in O(m),\ T_2 \in O(n),$$

and $\sigma_1 \geq \cdots \geq \sigma_m \geq 0$.

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**Theorem ([Breiding et al., 2024, Thm. 2.9])**

The closest matrix to $U$ in $\mathcal{M}_{r,m\times n} \cap \mathbb{R}^{m\times n}$ is

$$T_1 \cdot \mathrm{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0) \cdot T_2.$$

All complex critical points of the Euclidean distance are real.
They are found by substituting zeros for $m - r$ entries of $\mathrm{diag}(\sigma_1, \ldots, \sigma_m)$.

$$\implies \quad \mathrm{EDdeg}(\mathcal{M}_{r,m\times n}) = \binom{m}{r}.$$

# Linear Networks Invariant under a Cyclic Group Action

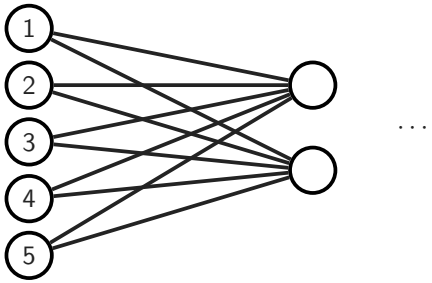



Figure: Weight sharing induced by the permutation of the input nodes (135)(24).

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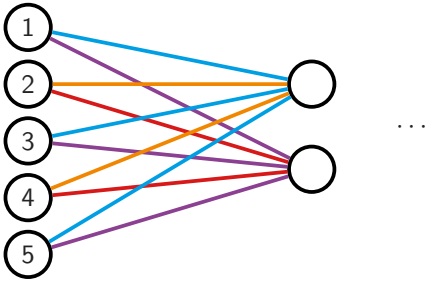



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Let $\mathcal{M}_{r,m\times n}$ be the $m \times n$-matrices of rank $\leq r$ (over $\mathbb{C}$).
Let $\pi$ be a permutation of $\{1, \ldots, n\}$ composed of $k$ cycles.
$\pi$ operates on $\mathbb{C}^n$ by interchanging entries.

$$\mathcal{M}^{\pi}_{r,m\times n} := \{M \in \mathcal{M}_{r,m\times n} \mid M \cdot \pi = M\}\,.$$

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$$\implies \mathrm{EDdeg}(\mathcal{M}^{\pi}_{r,m\times n}) = \tbinom{\min(m,k)}{\min(r,k)}.$$

# Convolutional Matrices

Fix a vector $(w_0, \ldots, w_{k-1}) \in \mathbb{C}^k$ (=*filter*).
A *convolutional matrix* is of the form

$$\begin{pmatrix} w_0 & w_1 & \cdots & w_{k-1} & 0 & \cdots & 0 \\ 0 & w_0 & w_1 & \cdots & w_{k-1} & 0 \cdots & 0 \\ & & \ddots & \ddots & & \ddots & \\ 0 & \cdots & 0 & w_0 & w_1 & \cdots & w_{k-1} \end{pmatrix} \in \mathbb{C}^{m \times n}, \quad \text{with } m = n - k + 1.$$

# Convolutional Matrices: Multiplication

One finds:

$$\begin{pmatrix} m_1 \times n_1 \\ \text{filter length: } k_1 \end{pmatrix} \cdot \begin{pmatrix} m_2 \times n_2 \\ \text{filter length: } k_2 \end{pmatrix} = \begin{pmatrix} m_1 \times n_2 \\ \text{filter length: } k_1 + k_2 - 1 \end{pmatrix}$$

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Example:

$$\begin{pmatrix} a & b \end{pmatrix} \cdot \begin{pmatrix} c & d & 0 \\ 0 & c & d \end{pmatrix} = \begin{pmatrix} a \cdot c & a \cdot d + b \cdot c & b \cdot d \end{pmatrix}.$$

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This multiplies like the following homogeneous polynomials (in $\mathbb{C}[x, y]$):

$$(a \cdot x + b \cdot y) \cdot (c \cdot x + d \cdot y) = a \cdot c \cdot x^2 + (a \cdot d + b \cdot c) \cdot x \cdot y + b \cdot d \cdot y^2.$$

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**[Kohn et al., 2025, Prop. 2.6]**

There is a general multiplicative correspondence between convolutional matrices and homogeneous polynomials in $\mathbb{C}[x, y]$.

# Multiple Root Loci

Identify points in $\mathbb{C}^k$ with the homogeneous polynomials $\mathbb{C}[x, y]_{k-1}$.
Fix an integer partition $\overrightarrow{\lambda} = (\lambda_1, \ldots, \lambda_r)$ of $k - 1$.

Consider the algebraic set:

$$\Delta_{\overrightarrow{\lambda}} := \{f \in \mathbb{C}[x, y]_{k-1} \mid \text{the proj. roots of } f \text{ have multiplicities } \lambda_1, \ldots, \lambda_r\} \subseteq \mathbb{C}^k$$

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- Generalization of discriminant hypersurfaces.
- Already studied by Hilbert [Hilbert, 1887].

# Linear Convolutional Networks

Linear Convolutional Networks (LCNs) arise as products of convolutional matrices.

$$\mathcal{C}_{k_1,\dots,k_N,m} := \{W_N \cdots W_1 \mid W_i \text{ conv. matrix with filter length } k_i\} \quad ,$$

where $m$ is the dimension of the input space.

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**Theorem ([Kohn et al., 2025, Thms. 5.2-5.4])**

$$\mathrm{EDdeg}(\mathcal{C}_{k_1,\dots,k_N,m}) = \max_{\overrightarrow{\lambda} \text{ part. of } k-1} \mathrm{EDdeg}(\Delta_{\overrightarrow{\lambda}}),$$

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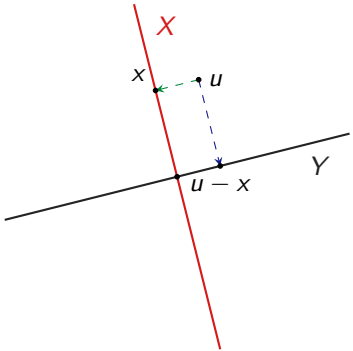
Sample computations of $\mathrm{EDdeg}(\Delta_{\overrightarrow{\lambda}})$ can be found e.g. in [Lee and Sturmfels, 2016].

# The Euclidean Distance Degree and Duality

# Dual Varieties

Fix an affine cone $X \subseteq \mathbb{C}^n$.

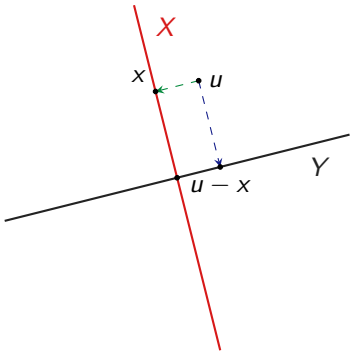
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**Theorem ([Draisma et al., 2016, 5.2])**

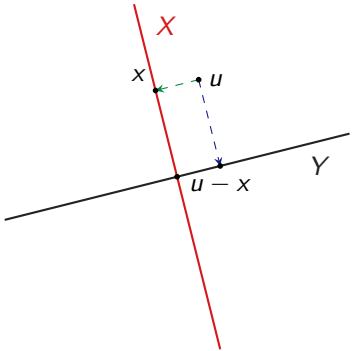
$x \mapsto u - x$ is a bijection between

- critical points of $d_u$ on $X$ and
- critical points of $d_u$ on $Y$.

# Dual Varieties

Fix an affine cone $X \subseteq \mathbb{C}^n$.

$$Y := \overline{\{y \in \mathbb{C}^n \mid \exists x \in X \setminus X_{\text{sing}} \colon y \perp T_x X\}}.$$




**Theorem ([Draisma et al., 2016, 5.2])**

$x \mapsto u - x$ is a bijection between

- critical points of $d_u$ on $X$ and
- critical points of $d_u$ on $Y$.

$\implies \text{EDdeg}(X) = \text{EDdeg}(Y)$.

## Catanese-Trifogli

Let $X \subseteq \mathbb{P}^{n-1}$ be smooth, irreducible, and of dimension $m$. Assume that $X \cap Z(x_1^2 + \cdots + x_n^2)$ is smooth.

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**Theorem ([Draisma et al., 2016, Thm. 5.8])**

$$\mathrm{EDdeg}(X) = \sum_{i=0}^{m} (-1)^i \cdot (2^{m+1-i} - 1) \deg(c_i(X)),$$

where $c_i(X)$ is the $i^{\text{th}}$ Chern class of the tangent bundle of $X$.

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The Euclidean distance degree of a subvariety $\subseteq \mathbb{P}^{n-1}$ is defined as the one of its cone $\subseteq \mathbb{C}^n$.

# A "Practical" Upper Bound on the Euclidean Distance Degree

Let $X$ be a variety of codimension $c$ in $\mathbb{C}^n$ defined by polynomials

$$f_1, f_2, \ldots, f_c, \ldots, f_s, \quad \text{with degrees } d_1 \geq d_2 \geq \cdots \geq d_c \geq \cdots \geq d_s.$$

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**Proposition ([Draisma et al., 2016, Prop. 2.6])**

$$\mathrm{EDdeg}(X) \leq d_1 d_2 \cdots d_c \cdot \sum_{i_1+i_2+\cdots+i_c \leq n-c} (d_1-1)^{i_1} \cdot (d_2-1)^{i_2} \cdots (d_c-1)^{i_c}.$$

Equality holds when $X$ is a general complete intersection of codimension $c$.

# Appendix

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